Pre-Calculus Exam (Year 112)

- Any device with a computer algebra system is prohibited during the exam.
- There are two sections in this exam. Please read the respective instruction very carefully.

Section A. (60%)

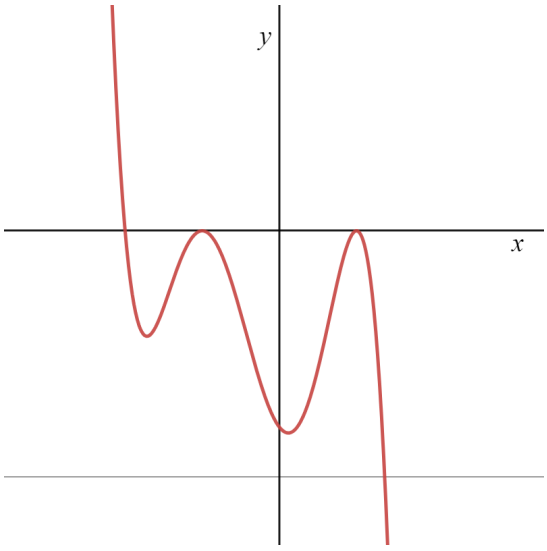
Instruction : For each question in this section, there is only ONE correct answer. Each correct answer is worth 3%.

1. $f(x)$ is a one-to-one function with domain $(0,1)$ and range $(-\infty,\infty)$. Which of the following statements is false?
   (A) $f(x)$ has an inverse function $f^{-1}(x)$ whose domain is $(-\infty,\infty)$.
   (B) We can define $f(g(x))$ for all $x \in (-\infty,\infty)$ where $g(x) = \dfrac{1}{1+x^2}$.
   (C) $f(f^{-1}(x)) = x$ for all $x \in (-\infty,\infty)$.
   (D) $f^{-1}(f(x)) = x$ for all $x \in (-\infty,\infty)$.
   (E) We can define $f^{-1}(f^{-1}(x))$ for all $x \in (-\infty,\infty)$.

2. Suppose that $f(x)$ is a function defined on $[-3,3]$. Which of the following statements is always true?
   (A) $f(x) - f(-x)$ is an even function defined on $[-3,3]$.
   (B) $f(x) + f(-x)$ is an odd function defined on $[-3,3]$.
   (C) If $f(x)$ is an even function, then $f(x)$ has no inverse function.
   (D) If $f(x)$ is an odd function, then $f(x)$ has no inverse function.
   (E) None of the above.

3. The graph of $f(x)$ is shifted to the right by 2 units and reflected with respect to the line $y = 3$. The resulting graph has equation:
   (A) $y = 6 - f(x-2)$
   (B) $y = 3 + f(x-2)$
   (C) $y = 6 - f(x+2)$
   (D) $y = 3 + f(x+2)$
   (E) None of the above.

4. Which one of the following equations could possibly have the graph given below?



   (A) $y = 2(x+2)(x+1)^2(x-1)^2$
   (B) $y = -2(x+2)(x+1)^2(x-1)^2$
   (C) $y = -2(x+2)^2(x+1)(x-1)^2$
   (D) $y = 2(x+2)^2(x+1)^2(x-1)$
   (E) $y = -2(x+2)^2(x+1)^2(x-1)$

5. For what value of k would the equation $x^2 + 3x + 4 = kx$ have two distinct real solutions?

(A) -1
(B) 3
(C) 5
(D) 7
(E) 9

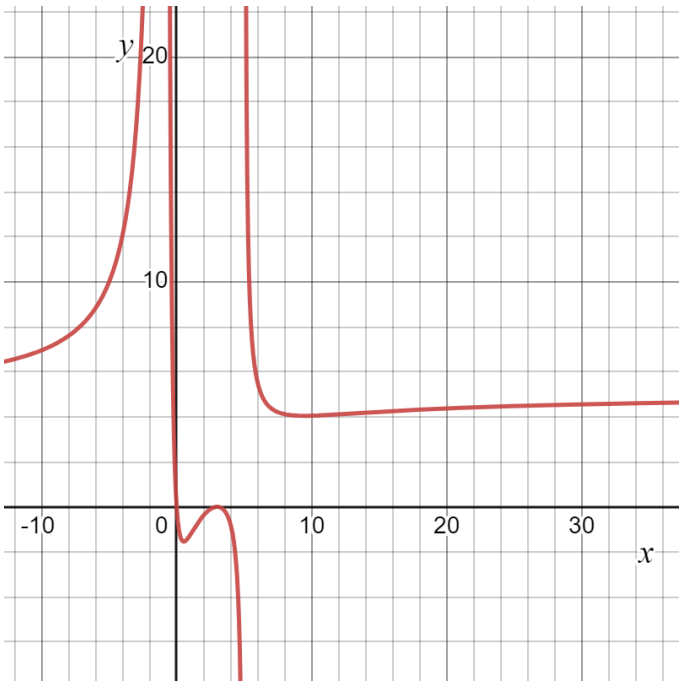
6. The maximum value M of the function $f(x) = 4 - x^2 - 6x + 2kx$ occurs at $x = x_0$. Find x_0 and M.

(A) $x_0 = 3 - k,\ M = 4 - (3 - k)^2$
(B) $x_0 = k - 3,\ M = 4 - (3 - k)^2$
(C) $x_0 = 2k - 3,\ M = 4 + (3 - 2k)^2$
(D) $x_0 = 3 - 2k,\ M = 4 + (3 + 2k)^2$
(E) $x_0 = k - 3,\ M = 4 + (3 - k)^2$

7. The graph of the function

$$f(x) = \frac{ax(x-3)^b}{(x+1)^c(x-5)}$$

is given as below. Find the possible constants a, b, c.



(A) $a = 1,\ b = 2,\ c = 2$
(B) $a = 5,\ b = 2,\ c = 2$
(C) $a = 10,\ b = 3,\ c = 3$
(D) $a = 5,\ b = 4,\ c = 2$
(E) $a = 5,\ b = 3,\ c = 3$

8. Which of the following statements is true about the polynomial $f(x) = x^6 + x^2 + 1$?

(A) Because $f(x) \neq 0$ for all $x \in \mathbb{R}$, $f(x)$ is irreducible.
(B) Because $f(x) \neq 0$ for all $x \in \mathbb{R}$, $f(x)$ has no complex root.
(C) Because $f(x) \neq 0$ for all $x \in \mathbb{R}$, $f(x)$ has no linear factor with real coefficients.
(D) $f(x)$ has no quadratic factors.
(E) $x^2 + 1$ is a factor of $f(x)$.

9. Consider $f(x) = 3 - \sqrt{-x^2 + x}$. Which of the following statements is true?

   (A) $f(x)$ is not defined for all $x \in \mathbb{R}$.

   (B) The graph of $f(x)$ is a circle centered at $\left(\frac{1}{2}.3\right)$.

   (C) The graph of $f(x)$ is a left half circle with radius $\frac{1}{4}$.

   (D) The graph of $f(x)$ is a lower half circle with radius $\frac{1}{2}$.

   (E) None of the above.

10. Find the interval of values of x that satisfies the inequality $\frac{x}{x+2} \geq 3$.

   (A) $x \in [-3, -2)$

   (B) $x \in (-\infty, -3]$

   (C) $x \in [-3, \infty)$

   (D) $x \in (2, \infty)$

   (E) None of the above.

11. Suppose $0 \leq x < 2\pi$. How many solutions does the equation $7 \sin x + 2 \cos^2 x = 5$ have ?

   (A) 0

   (B) 1

   (C) 2

   (D) 3

   (E) 4

12. Which of the following numbers is the *smallest* ?

   (A) $\ln 2$

   (B) $\sqrt{\ln(2^2)}$

   (C) 1

   (D) $\left(\frac{1}{\ln 2}\right)^3$

   (E) $\frac{1}{\ln \sqrt{2}}$

13. Find the value of $\sin \frac{3\pi}{4} + \cos \frac{\pi}{4} + \tan \frac{5\pi}{4} + \cot \frac{7\pi}{4}$.

   (A) $2 + \sqrt{2}$

   (B) $\sqrt{2}$

   (C) $\sqrt{2} - 2$

   (D) 2

   (E) None of the above

14. If $\cos(x) = \frac{3}{5}$, then the value of $\cos(2x)$ is

   (A) $\frac{18}{25}$

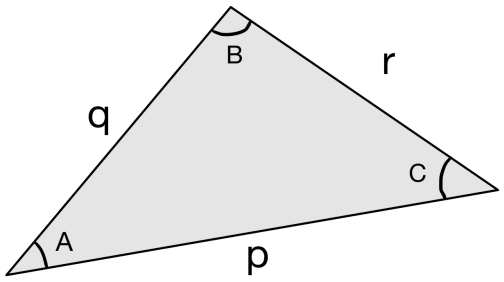
   (B) $\frac{7}{25}$

   (C) $-\frac{7}{25}$

   (D) $-\frac{18}{25}$

   (E) not determined uniquely, depending on the value of x.

15. Which of the following equations correctly describes a relation between the lengths p, q, r and the angles A, B, C of the following triangle ?



(A) $\dfrac{\sin A}{p} = \dfrac{\sin B}{q} = \dfrac{\sin C}{r}$

(B) $\dfrac{\sin B}{r} = \dfrac{\sin C}{p} = \dfrac{\sin A}{q}$

(C) $\cos(B) = \dfrac{p^2 + q^2 - r^2}{2pq}$

(D) $\cos(C) = \dfrac{p^2 + r^2 - q^2}{2pr}$

(E) None of the above.

16. Let $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$. Simplify the expression

$$(((\mathbf{i} \times \mathbf{j}) \times \mathbf{j}) \times \mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}).$$

(A) $\langle 0, 0, -7\rangle$

(B) $\langle 0, 0, 0\rangle$

(C) -7

(D) 0

(E) 7

17. Consider four points $P = (1, 1, 0)$, $Q = (2, 0, 3)$, $R = (0, 2, 5)$, $S = (-1, 3, 2)$. Determine which of the following statement is correct.

(A) $\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{PS}$

(B) $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{RS}$

(C) $\overrightarrow{PR}$ is parallel to $\overrightarrow{QS}$

(D) $PQRS$ is a parallelogram (平行四邊形).

(E) None of the above.

18. Consider the vectors $\vec{u} = \langle 1, 1, 1\rangle$ and $\vec{v} = \langle -1, 2, 1\rangle$. Let α be the acute angle between $\vec{u}$ and $\vec{v}$. Find the value of $\cos(\alpha)$.

(A) $\dfrac{\sqrt{3}}{2}$

(B) $\dfrac{\sqrt{2}}{3}$

(C) $\dfrac{1}{3}$

(D) $\dfrac{1}{\sqrt{2}}$

(E) None of the above.

19. Find the coefficient (係數) of x^7 of the polynomial $f(x) = \sum_{k=3}^{15} (-1)^k \frac{x^{k-2}}{(4k-1)}$.

(A) $-\frac{1}{27}$

(B) $\frac{1}{31}$

(C) $-\frac{1}{35}$

(D) $\frac{1}{39}$

(E) None of the above.

20. Find the sum $\sum_{n=15}^{150} (4n+1)$.

(A) 16054

(B) 45016

(C) 56014

(D) 60145

(E) None of the above

Section B. (40%)

Instruction : For each question in this section, choose all correct answers. For each question,

- if you make no mistakes in your choices, you get 4%;
- if you make one mistake in your choices, you get 3%;
- if you make two mistakes in your choices, you get 2%;
- if you make more than two mistakes in your choices, you get 0%;

21. Let $f(x) = \log_4(8x+2)$. Determine which of the following statements is/are correct.

(A) The domain of $f(x)$ is $\left(-\frac{1}{4}, \infty\right)$.

(B) $f(x) = \frac{1}{2}\left[3 + \log_2\left(x + \frac{1}{4}\right)\right]$.

(C) The graph of $f(x)$ is obtained by shifting the graph of $\log_2 x$ to the right by $\frac{1}{4}$ units and then shifting upward by $\frac{3}{2}$ units.

(D) The inverse function of $f(x)$ is $f^{-1}(x) = \frac{1}{8}(4^x - 2)$.

(E) $f(x) \geq \log_2 x$ has no real solutions.

22. In which of the following intervals do we have

$$\frac{2(x^2+1)^2 - 5x(x^2+1)}{x^2(3-x)} \geq 0 \ ?$$

(A) $(-\infty, 0)$

(B) $\left(0, \frac{1}{2}\right)$

(C) $\left(\frac{1}{2}, 2\right)$

(D) $(2, 3)$

(E) $(3, \infty)$

23. In which of the following intervals do we have

$$x^2 + x + 1 \leq \frac{5x^2 - 6x - 1}{x - 1} \ ?$$

(A) $(-3, -2)$

(B) $(0, 2)$

(C) $(2, 3)$

(D) $(3, 4)$

(E) $\left(\frac{1}{2}, 1\right)$

24. Let $f(x) = |x - 3|$ and $g(x) = \sqrt{4x^2 - 4x + 1}$. Which of the following statements is/are correct?

(A) $f(g(x))$ is defined for all $x \in \mathbb{R}$.

(B) $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$.

(C) $f(g(x)) = 2x - 4$ for $x > 2$.

(D) $f(g(x)) = 4 - 2x$ for $x < 2$.

(E) $f(g(x)) = 2 + 2x$ for $-1 \leq x \leq 0$.

25. There are two statements about a real number x :

P : $\dfrac{x^2 - 1}{x + 2} < 0$,

Q : $\dfrac{x}{1 - x} > a$.

Assume that Q implies P. Choose possible values of a.

(A) $a = 2$

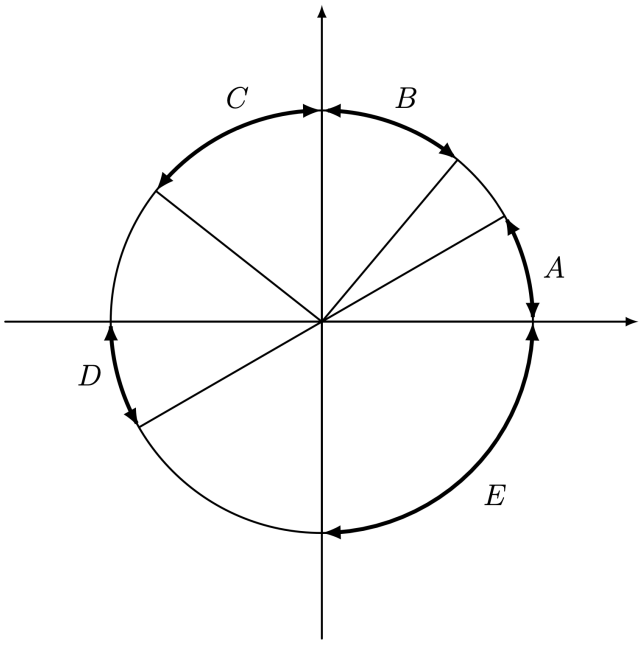
(B) $a = 1$

(C) $a = 0$

(D) $a = -\dfrac{1}{3}$

(E) $a = -1$

26. The picture below shows the unit circle, where each point has coordinates $(\cos x, \sin x)$ for some angle x. Determine among which of the following ranges do we have $\sin x < \cos x$.



27. Determine which of the following are solutions to the equation

$$4\sin^3 x - 8\sin^2 x - \sin x + 2 = 0.$$

(A) $\dfrac{\pi}{6}$

(B) $\dfrac{3\pi}{4}$

(C) $\dfrac{5\pi}{6}$

(D) $\dfrac{7\pi}{4}$

(E) $\dfrac{11\pi}{6}$

28. Determine which of the following statements is/are correct.

(A) The planes $x + 2y + 3z = 5$ and $x - 2y + z = 5$ are perpendicular to each other.

(B) The vector $\mathbf{v} = \langle 1, 2, 3\rangle$ is perpendicular to the plane $x - 2y + z = 5$.

(C) The equation of the tangent line to $y = e^x$ at $(0, 1)$ is $y - x = 1$.

(D) If $a > 1$, the function $f(x) = a^x$ has an inverse function.

(E) If $0 < a < 1$, the function $f(x) = a^x$ has an inverse function.

29. Determine which of the following statements is/are correct.

(A) For any $a, b > 0$, we have $\ln(\sqrt{a+b}) = \dfrac{1}{2}(\ln a + \ln b)$.

(B) The domain of $f(x) = \ln(x^2 - 2x + 2)$ is $\mathbb{R}$ (all real numbers).

(C) The range of $f(x) = \ln(x^2 - 2x + 2)$ is $\mathbb{R}$ (all real numbers).

(D) $\log_2(3) \cdot \log_5(2) = \log_5(3)$.

(E) The equation $\ln(x) - \ln(3x + 1) = 1$ has no real solutions.

30. A sequence $\{a_k\}_{k=0}^{\infty}$ is defined by $a_0 = p$ and $a_k = (a_{k-1})^2$ for $k > 0$, with $0 < p < 1$. The sequence $\{b_k\}_{k=0}^{\infty}$ is defined by $b_k = \ln(a_k)$. Determine which of the following statements is/are correct.

(A) $\{\sqrt{a_k}\}_{k=0}^{\infty}$ is a geometric sequence.

(B) $b_k \geq 0$ for every $k \geq 0$.

(C) $\{|b_k|\}_{k=0}^{\infty}$ is an arithmetic sequence.

(D) $\{b_k\}_{k=0}^{\infty}$ is a geometric sequence.

(E) $\displaystyle\sum_{k=0}^{n} b_k = (2^{n+1} - 1) \cdot \ln(p)$.